

New version of pseudo-Hermiticity in the two-sided deformation of Heisenberg algebra

A.M. Gavrilik[#] and I.I. Kachurik^{#‡}

January 22, 2016

[#] Bogolyubov Institute for Theoretical Physics of NAS of Ukraine
14b, Metrologichna Str., Kyiv 03680, Ukraine

[‡] Khmelnytskyi National University
11, Instytutska Str., 29016 Khmelnytskyi, Ukraine

Abstract

The recently introduced two- and three-parameter (p, q) - and (p, q, μ) -deformed extensions of the Heisenberg algebra were explored under the condition of their connectedness with the respective nonstandard (other than known ones) deformed quantum oscillator algebras. In this paper we show that such connection dictates certain new $\eta(N)$ -pseudo-Hermitian conjugation rule between the creation and annihilation operators, with $\eta(N)$ depending on the particle number operator N . In turn, that leads to the related $\eta(N)$ -pseudo-Hermiticity of the position/momentum operators, though the involved Hamiltonian is Hermitian. Different possible cases are studied, and some interesting features implied by the use of such $\eta(N)$ -based conjugation rule are emphasized.

Keywords: deformed Heisenberg algebra; position and momentum operators; deformed oscillator algebra; $\eta(N)$ -pseudo-Hermitian conjugation; $\eta(N)$ -pseudo-Hermiticity.

PACS: 03.65.-w; 03.65.Fd; 02.20.Uw; 05.30.Pr; 11.10.Lm

1 Introduction

Non-Hermitian modifications of quantum mechanics [1–12] which lead nevertheless to real spectra of operators, attract great interest. A significant part of such investigations in recent years has been crystalized into an important branch encompassing the works on pseudo-Hermitian [5] representation in quantum mechanics, see the comprehensive review [10] which gives a plenty of references and discusses main ideas and results. This approach have its impact on a variety of applications, ranging from nuclear physics and quantum field theory to nonlinear optics and biophysics [10].

On the other hand, significant attention is devoted to generalized versions [12–20] of Heisenberg algebra (HA), obtained through appropriate extension of its basic relation $[X, P] = i\hbar$. That implies respective modifications of the uncertainty relation (see e.g. [15, 16, 21–24]). In our recent paper [20], the so-called two-sided or three-parameter (p, q, μ) -deformation of HA has been introduced and studied. Its particular $p = 1, \mu = 0$ case yields simple modified HA with q -commutator involved which was studied in [17], where the explicit relation with certain non-standard q -deformed oscillator algebra was found (for some recent applications of deformed oscillators or deformed bosons see e.g. [25–32]). In the more general ”left-handed” + ”right-handed” generalization [20], the analogous mapping onto deformed oscillator has been derived in the case of p, q -deformation of HA (involving p, q -commutator) as well as for the case of three-parameter p, q, μ -deformed HA, where the related non-standard deformed oscillator algebra (DOA) was obtained in a somewhat restricted situation. In conjunction with the mentioned relation, an important property was deduced that the deformation parameter μ and also the parameters p and q explicitly depend on the particle number operator N .

In all three mentioned cases, i.e. q - , (p, q) - and (p, q, μ) -deformations of HA, the formulas relating the position and momentum operators X, P and creation/destruction operators a^+, a^- are not those of usual harmonic oscillator, see e.g. [33], but involve N -dependent coefficients. This fact, rooted in the imposed condition of realizability of particular deformed HA through respective DOA, is of basic importance as it dictates principal distinction from the usual (Hermitian) conjugation rules of the operators involved. However, such rules were not considered in [20].

Therefore, the goal of the present paper is to examine those aspects of 2- and 3-parameter deformed HA (or DHA) which concern modified rules of (self)conjugation of the operators involved. In our study, most important aspect is the encountered unusual $\eta(N)$ -pseudo-Hermitian conjugation and the related $\eta(N)$ -pseudo-Hermiticity of X and/or P . About the plan of our paper: Sections 2-5 give a sketch of those deformed versions of HA which serve as playground. The reasons of why, instead of usual Hermitian conjugation and usual Hermiticity, there inevitably emerges the concept of $\eta(N)$ -pseudo-Hermitian conjugation, along with $\eta(N)$ -pseudo-Hermiticity, are described in Sec. 6. Therein, we also study the cases with partial $\eta(N)$ -pseudo-Hermiticity (when one of the operators, X or P , remains Hermitian). In Sec. 7 we show that both X and P should obey $\eta(N)$ -pseudo-Hermiticity even in the case when a^+ and a^- are usual Hermitian conjugates of each other. General situation when $\eta(N)$ -pseudo-Hermitian conjugation concerns all the four operators is treated in Sec. 8. Next 9th Section deals with the properties of how X, P commute with the particle number operator N , while the Hamiltonian (in terms of X, P) and its Hermiticity are the subject of Sec. 10. The paper ends with concluding remarks.

2 Extended Heisenberg algebra with Hamiltonian or P^2 in R.H.S.

The Heisenberg algebra (HA), based on the well-known relation of commutation

$$[X, P] = i\hbar , \tag{1}$$

during last decades serves as starting point for diverse modifications or generalizations. Rather general and one of most natural extensions of (1) involves in its r.h.s. a function $f(\mathcal{H})$ of the Hamiltonian \mathcal{H} , that is

$$[X, P] = i\hbar f(\mathcal{H}) . \quad (2)$$

Some versions of this modification of HA were studied e.g. in [13, 14, 19], exploiting either $f(P^2, X^2)$, or $f(P^2)$, or the particular form $\exp(\kappa P^2)$. With constant term only (or zeroth order in \mathcal{H}), the relation (2) reduces to the customary HA (1).

An important special case of Eq. (2), namely the algebra based on the relation

$$[X, P] = i\hbar(1 + \mu\mathcal{H}) , \quad \mu \in \mathbb{R}, \quad (3)$$

was explored in [13] with the impact on quantum mechanics at the extreme conditions of high energy physics and quark physics. This line of research was developed in a number of papers such as [12, 14–16, 19, 20] and others.

3 A p, q -deformed Heisenberg algebra: the connection with DOA

Another approach to deform HA affects the l.h.s. of defining relation, and yields the two-parameter or p, q -deformation of the form

$$pXP - qPX = i\hbar , \quad (4)$$

introduced and studied in Ref. 20. Note that its special case $p = 1$ was earlier analyzed by Chung and Klimyk in Ref. [17].

In Ref. [20], main goal was to connect the deformed HA (4), and its two-sided 3-parameter extension, with an appropriate DOA. As explained in many papers, see e.g. the overviews [34, 35], each version of DOA is generated by three generating elements a^+ , a^- and N (with physical meaning respectively the creation, the annihilation, and the excitation number operators, within the Fock type representation). The three generators of DOA obey the following defining relations:

$$[N, a^\pm] = \pm a^\pm, \quad a^+ a^- = \phi(N), \quad a^- a^+ = \phi(N + 1), \quad (5)$$

and the generalization

$$a^- a^+ - a^+ a^- = \phi(N + 1) - \phi(N) \quad (6)$$

of usual commutation relation $[a^-, a^+] = 1$ (this is recovered at $\phi(N) = N$). Here $\phi(N)$ is the *deformation structure function* (DSF): indeed, its particular form completely defines the corresponding DOA. Moreover, in the Fock type representation (and basis) the DSF $\phi(N)$ explicitly determines matrix elements of the operators a^+ and a^- according to the formulas

$$N|n\rangle = n|n\rangle, \quad a^+|n\rangle = \sqrt{\phi(n+1)}|n+1\rangle, \quad a^-|n\rangle = \sqrt{\phi(n)}|n-1\rangle.$$

These explain the names "creation, annihilation, and number" operators.

The procedure of connecting the DHA given by (4) with an appropriate DOA was described in detail in Ref. [20], so we do not need to reproduce its full content here.

However, some of the results necessary for our present goals will be recalled below. Let us first note that there exists yet another form of generalized commutation relation for a^- and a^+ , namely

$$G(N)a^-a^+ - H(N)a^+a^- = 1. \quad (7)$$

As known[34], given the latter form one can go over to the form (6) above involving the DSF $\phi(N)$.

To find $G(N)$ and $H(N)$ we assume the relation between X , P appearing in (4), and the triple a^- , a^+ and N , generating DOA, in rather general form

$$X = f(N)a^- + g(N)a^+, \quad P = i(k(N)a^+ - h(N)a^-).$$

Using the latter jointly with relations (4) and (7), one is able to find explicitly (see Ref. [20] for more details) first the functions $f(N)$, $g(N)$, $h(N)$, $k(N)$ and then, by means of these, also the functions $G(N)$ and $H(N)$:

$$\begin{aligned} f(N) = k(N) &= \frac{1}{\sqrt{2}} Q^N, & h(N) = g(N) &= \frac{1}{\sqrt{2}} Q^{2N}, & Q &\equiv q/p; \\ H(N) &= \frac{1}{2} q Q^{2N} (1 + Q^{2N+2}), & G(N) &= \frac{1}{2} p Q^{2N} (1 + Q^{2N-2}). \end{aligned}$$

Thus, we infer the important fact that X and P in terms of the creation, annihilation, and particle number operators are expressed as

$$X = \frac{1}{\sqrt{2}} [Q^{2N}a^+ + Q^Na^-], \quad P = \frac{i}{\sqrt{2}} [Q^Na^+ - Q^{2N}a^-], \quad (8)$$

which will be used in Sections 6-8. The inverse relations readily follow, so that

$$a^- = d_{N,Q}(Q^{-N}X + iP), \quad a^+ = d_{N,Q}(X - iQ^{-N}P), \quad d_{N,Q} \equiv \sqrt{2}(1 + Q^{2N})^{-1}. \quad (9)$$

Obviously, the restriction $Q = 1$ implies $d_{N,1} = \frac{1}{\sqrt{2}}$ and brings us back to the well-known linear relations between a^+ , a^- and X , P (see e.g. [33]).

4 Skew-Hermiticity of the basic relation (4)

Here we examine consistency of the basic relation (4) from the viewpoint of conjugation: since r.h.s. of (4) is skew-Hermitian, the same property should be valid for the l.h.s. To this end, consider the cases of real and complex p, q separately.

(A) Let $p, q \in \mathbb{R}$.

Assume that $X^\dagger = X$ and $P^\dagger = P$. Then, skew-hermiticity of the l.h.s. of (4) does holds only if $p = q$. This case however is not interesting for us as it reduces to the non-deformed one for the operators \tilde{X} and \tilde{P} such that $\tilde{X} \equiv \sqrt{q}X$ and $\tilde{P} \equiv \sqrt{q}P$.

Now let $P^\dagger = P$, but $X^\dagger = \kappa X \neq X$ with constant κ . By demanding skew-hermiticity of the l.h.s. of (4) we deduce:

$$(\kappa p - q)PX + (p - \kappa q)XP = 0.$$

For $PX \neq 0$, it must be that either (i) $q = \kappa p$ and $p = \kappa q$ which implies $\kappa^2 = 1$ i.e. $\kappa = \pm 1$ (the first option is trivial and the second one is unphysical), or (ii) there should be $XP = \omega PX$ where $\omega = \frac{\kappa p - q}{\kappa q - p}$. Using (4) we infer that $PX = i\hbar I/(p\omega - q)$ which means P is proportional to inverse of X , which is also rather exotic.

Same conclusion is drawn if $X^\dagger = X$ and $P^\dagger = \kappa P$, or if $X^\dagger = \kappa X$ and $P^\dagger = \kappa' P$.

More general case involves $P^\dagger = P$ and $X^\dagger = \tilde{\eta}X\tilde{\eta}^{-1}$ (i.e. the operator X is pseudo-Hermitian), and the case with both $P^\dagger = \eta'P(\eta')^{-1}$ and $X^\dagger = \tilde{\eta}X\tilde{\eta}^{-1}$ (the two operators are pseudo-Hermitian). This will be considered below, see Sec. 7.

(B) Let $p, q \in \mathbb{C}$.

With same assumption that $X^\dagger = X$ and $P^\dagger = P$, we infer:

$$(\bar{p} - q)PX + (p - \bar{q})XP = 0.$$

For $XP \neq 0$, we have that either

- (i) $p = \bar{q} = re^{-i\theta}$ and thus $e^{-i\theta}XP - e^{i\theta}PX = \frac{i\hbar}{r}$, or
- (ii) $p \neq \bar{q}$ and then $PX = \frac{\bar{q}-p}{\bar{p}-q}XP$, or equivalently $[P, X]_{\tilde{Q}} = 0$ where $[A, B]_s \equiv AB - sBA$ and $\tilde{Q} \equiv \frac{\bar{q}-p}{\bar{p}-q} = -\frac{p-\bar{q}}{(p-\bar{q})^*} = -e^{2i \arg(p-\bar{q})}$.

The found restrictions concern parameters p, q . In general (and more realistic in presence of deformation) case we will deal with pseudo-Hermitian X and/or P .

5 Two-sided (or three-parameter) deformed Heisenberg algebra

The two-sided, 3-parameter deformed extension of HA recently introduced in Ref. [20] combines different modifications of the HA (1) that yields

$$pXP - qPX = i\hbar(1 + \mu\mathcal{H}). \quad (10)$$

Again it is linked with certain deformed boson algebra such that the two relations

$$\tilde{H}(N)a^-a^+ - \tilde{G}(N)a^+a^- = 1, \quad a^-a^+ - a^+a^- = \tilde{\phi}(N+1) - \tilde{\phi}(N) \quad (11)$$

are valid, where $\tilde{\phi}(N)$ is the respective structure function of deformation [34, 35]. This DSF was derived, in terms of the found $\tilde{H}(n)$ and $\tilde{G}(n)$, for two important cases [20]:

- (i) If $\mu = 0$ is set in (10) (turning it into (4)), the proper DSF in (11) results as

$$\begin{aligned} \tilde{\phi}(n) &= \frac{2p^{-1}Q^{-n}}{(1 + Q^{2n-2})(1 + Q^{2n})} \left(1 + \frac{Q^n - Q^{-n+1}}{Q - 1} \right) = \\ &= \frac{2q^{-n}p^{5n-3}}{(q^{2n-2} + p^{2n-2})(q^{2n} + p^{2n})} \left(1 + \frac{[2n-1]_{q,p}}{(qp)^{n-1}} \right) \end{aligned} \quad (12)$$

(here $[m]_{q,p} \equiv \frac{q^m - p^m}{q - p}$ denotes the q, p -number corresponding to a number m), and
(ii) for $\tilde{H}(N) = \tilde{G}(N)$ at $p \neq q$ we obtain (denote $Q = p/q$):

$$\check{\phi}(n) = \frac{4Q^2}{p(1+Q^2)(1+Q^3)} - \frac{4}{p(1+Q)} \left(\frac{1-Q^{2-2n}}{1-Q^2} + \sum_{j=1}^{n-1} \frac{1+Q^5}{Q^2(1+Q) + Q^{2j}(1+Q^5)} \right).$$

Recall that each DSF (e.g. $\tilde{\phi}(N)$) relates a^+a^- , a^-a^+ and N according to formulas

$$a^+a^- = \tilde{\phi}(N), \quad a^-a^+ = \tilde{\phi}(N+1),$$

and determines the corresponding action formulas for a^+ , a^- in the normalized basis of deformed analog [35] of Fock space so that

$$a^\pm |n\rangle = \sqrt{\tilde{\phi}\left(n + \frac{1 \pm 1}{2}\right)} |n \pm 1\rangle, \quad |n\rangle = (\tilde{\phi}(n)!)^{-\frac{1}{2}} (a^+)^n |0\rangle, \quad a^- |0\rangle = 0,$$

where $\tilde{\phi}(n)! = \tilde{\phi}(n) \tilde{\phi}(n-1) \dots \tilde{\phi}(2) \tilde{\phi}(1)$.

Formula (12) gives the DSF of nonstandard two-parameter deformed quantum oscillator. Nonstandard means it is nonsymmetric under $q \leftrightarrow p$ because of the factor $q^{-n}p^{5n-3}$ in the numerator. Thus it obviously differs from the well-known q, p -oscillator [36] whose structure function $\varphi_{q,p}(n) = [n]_{q,p}$ is $(q \leftrightarrow p)$ -symmetric.

Let us note that formulas (8)-(9) and the conclusions in the preceding Section about skew-Hermiticity extend to the two-sided deformation of HA, see Eq. (10), under the condition that μ is real and the Hamiltonian \mathcal{H} is Hermitian (the Hermiticity of \mathcal{H} is discussed in Sec. 9 below).

6 An $\eta(N)$ -pseudo-Hermitian conjugation of the operators a^+ , a^-

In this Section, two distinct cases will be considered.

Case A. Assume, at Hermitian N , the Hermiticity for the momentum operator

$$P^\dagger = P, \tag{13}$$

and then infer conjugation rules for a^\pm . From Eq. (13), using Eq. (8) we have

$$P^\dagger = \frac{-i}{\sqrt{2}} \left[(a^+)^\dagger Q^N - (a^-)^\dagger Q^{2N} \right] = \frac{i}{\sqrt{2}} \left[Q^N a^+ - Q^{2N} a^- \right] = \frac{i}{\sqrt{2}} \left[a^+ Q^{N+1} - a^- Q^{2N-2} \right]$$

that yields:

$$(a^+)^\dagger Q^N = a^- Q^{2N-2}, \quad (a^-)^\dagger Q^{2N} = a^+ Q^{N+1}.$$

From this, using the relation (18) below, we infer the new conjugation rules:

$$(a^+)^\dagger = \eta(N) a^-, \quad (a^-)^\dagger = a^+ \eta^{-1}(N), \quad \eta(N) \equiv Q^{N-1}. \tag{14}$$

We call this new kind of conjugation $\eta(N)$ -pseudo-Hermitian conjugation: it generalizes to $\eta = \eta(N)$ the known η -pseudo-Hermitian conjugation [10–12] (note that

η in those papers depended on the momentum P). Thus, a^+ and a^- are mutual $\eta(N)$ -pseudo-Hermitian conjugates of each other.

It is clear that instead of (14) we can also adopt the equivalent definition of $\eta(N)$ -pseudo-Hermitian conjugation, namely

$$(a^+)^\dagger = a^- \eta(N), \quad (a^-)^\dagger = \eta^{-1}(N) a^+, \quad \eta(N) \equiv Q^{N-2}. \quad (15)$$

Obviously, when $Q \rightarrow 1$ (i.e. at $p = q$), the both versions of $\eta(N)$ -pseudo-Hermitian conjugation, (14) and (15), go over into the usual Hermitian mutual conjugation of a^+ and a^- . We stress that this concerns the p, q differing from unity, as well as when the both are equal to 1.

Remark 1. With account of (14), we have usual Hermiticity for the bilinears,

$$(a^+ a^-)^\dagger = (a^-)^\dagger (a^+)^\dagger = a^+ \eta^{-1}(N) \eta(N) a^- = a^+ a^-, \quad (16)$$

$$(a^- a^+)^\dagger = (a^+)^\dagger (a^-)^\dagger = \eta(N) a^- a^+ \eta^{-1}(N) = a^- a^+, \quad (17)$$

where at the last step in (17) the permutation rule

$$\mathcal{F}(N) a^\pm = a^\pm \mathcal{F}(N \pm 1), \quad (18)$$

stemming from the first relation in (5) and valid for general function $\mathcal{F}(N)$, has been utilized. The same is true if one takes (15).

Remark 2. Instead of (14) (resp. (15)) we of course could take the standard, as for operators, shape of mutual conjugation, i.e. $(a^+)^\dagger = \zeta(N) a^- \zeta^{-1}(N)$ and $(a^-)^\dagger = \zeta(N) a^+ \zeta^{-1}(N)$. However in view of (18), after redefinition $\zeta(N-1) \zeta^{-1}(N) \rightarrow \eta(N)$ (resp. $\zeta(N) \zeta^{-1}(N+1) \rightarrow \eta(N)$), that would reduce to (14) (resp. (15)).

Pseudo-Hermiticity of the position operator X

Recall that $\eta(N)$ -pseudo-Hermitian conjugation (14) of a^+ and a^- has been inferred in view of the requirement (13). On the other hand, the property (14) causes non-Hermiticity of the operator X . That is, we have to modify conjugation rule for the operator X . So let us find the modified rule of self-conjugation for the position operator in the assumed form $X^\dagger = \tilde{\eta}^{-1}(N) X \tilde{\eta}(N)$. From (14) we have

$$X^\dagger = \frac{1}{\sqrt{2}} \left((a^+)^\dagger Q^{2N} + (a^-)^\dagger Q^N \right) = \frac{Q}{\sqrt{2}} \left(a^+ + Q^{3N} a^- \right),$$

and it can be easily verified that

$$X^\dagger = \tilde{\eta}^{-1}(N) X \tilde{\eta}(N) \quad \text{with} \quad \tilde{\eta}(N) = Q^{N^2}. \quad (19)$$

The same does follow if we take the rule of $\eta(N)$ -conjugation in the form (15).

Thus, for the conjugation properties of the momentum and position operators here we have usual *Hermiticity* of P , but the $\tilde{\eta}(N)$ -pseudo-Hermiticity of X , i.e.

$$P^\dagger = P \quad \text{and} \quad X^\dagger = Q^{-N^2} X Q^{N^2}. \quad (20)$$

That is certainly linked with the rule of $\eta(N)$ -pseudo-Hermitian mutual conjugation for a^+ and a^- given by (14) or (15).

Similar analysis can be carried out if one exchanges the roles of X and P .
Case B. This time let us require that

$$X^\dagger = X. \quad (21)$$

Then we are led to the conjugation rule

$$(a^+)^\dagger = \hat{\eta}(N)a^-, \quad (a^-)^\dagger = a^+\hat{\eta}^{-1}(N), \quad \hat{\eta}(N) = Q^{-N-2}. \quad (22)$$

As a consequence we arrive at $\tilde{\eta}(N)$ -pseudo-Hermiticity of P , i.e.

$$P^\dagger = \tilde{\eta}^{-1}(N) P \tilde{\eta}(N) \quad \text{with} \quad \tilde{\eta}(N) = Q^{-N^2}. \quad (23)$$

Thus, for the (self)conjugation rules for momentum/position operators in this case we have usual *Hermiticity* of X jointly with $\tilde{\eta}(N)$ -pseudo-*Hermiticity* of P :

$$X^\dagger = X \quad \text{and} \quad P^\dagger = Q^{N^2} P Q^{-N^2}, \quad (24)$$

the both linked with $\hat{\eta}(N)$ -pseudo-*Hermitian conjugation* of a^+ and a^- in (22).

It is interesting to compare the coordinated couple of conjugation rules (14) and (20), with the respective coordinated couple of conjugation rules (22) and (24).

Remark 3. It should be stressed that the $\eta(N)$ -dependence in the conjugation rules for a^+ and a^- , and the related $\eta(N)$ -self-conjugation of X and/or P , are rooted in the basic connection established in Ref. 20 : DHA \Leftrightarrow DOA (i.e. deformed Heisenberg algebra \Leftrightarrow deformed oscillator algebra).

7 The case when a^+ and a^- are usual conjugates of each other

Now let us require for a^+ and a^- the usual conjugation property: $(a^\pm)^\dagger = a^\mp$. Then it is easy to see that both $X^\dagger \neq X$ and $P^\dagger \neq P$. Therefore we consider these operators as $\eta(N)$ -pseudo-Hermitian ones, by imposing

$$X^\dagger = \eta_X^{-1}(N) X \eta_X(N), \quad P^\dagger = \eta_P^{-1}(N) P \eta_P(N) \quad (25)$$

where $\eta_X(N)$ and $\eta_P(N)$ are some functions of N possessing their corresponding inverses. To find $\eta_X(N)$ and $\eta_P(N)$ explicitly, we use the formulas (8) for X and P . Then, by a simple algebra we deduce the following recurrence relations:

$$\eta_X(N+1) = \eta_X(N) Q^{N+2}, \quad \eta_P(N+1) = \eta_P(N) Q^{-N+1}.$$

Solving them we find respectively

$$\eta_X(N) = Q^{\frac{1}{2}N(N+3)} \eta_X(0), \quad \eta_P(N) = Q^{\frac{1}{2}N(-N+3)} \eta_P(0).$$

Obviously, the convenient choice is to set $\eta_X(0) = \eta_P(0) = 1$.

8 On the $\eta(N)$ -pseudo-Hermitian conjugation of a^\pm , X and P

To consider most general situation when the rules of pseudo-Hermitian conjugation concern both the pair a^+ , a^- and the operators X , P , we impose the relations

$$(a^+)^\dagger = \eta_a(N) a^-, \quad (a^-)^\dagger = a^+ \eta_a^{-1}(N), \quad (26)$$

$$X^\dagger = \eta_X^{-1}(N) X \eta_X(N), \quad P^\dagger = \eta_P^{-1}(N) P \eta_P(N), \quad (27)$$

where all the three η 's are different.

We wish to find relations governing the η 's. For this, we take conjugate X^\dagger of X in (8), then use (26) and compare with X^\dagger in (27). That results in the equations

$$Q^{2N+2} \eta_a(N) = Q^N \frac{\eta_X(N+1)}{\eta_X(N)}, \quad \frac{Q^{N-1}}{\eta_a(N-1)} = \frac{Q^{2N} \eta_X(N-1)}{\eta_X(N)},$$

or equivalently in the equations

$$\eta_a(N) = Q^{-N-2} \frac{\eta_X(N+1)}{\eta_X(N)}, \quad \frac{1}{\eta_a(N-1)} = \frac{Q^{N+1} \eta_X(N-1)}{\eta_X(N)}. \quad (28)$$

The latter two are not independent, being inverse of each other (shift $N \rightarrow N+1$).

Likewise, taking conjugate of P in (8), then using (26) and comparing with P^\dagger in (27), we obtain the equations

$$Q^{N+1} \eta_a(N) = Q^{2N} \frac{\eta_P(N+1)}{\eta_P(N)}, \quad \frac{Q^{2N-2}}{\eta_a(N-1)} = \frac{Q^N \eta_P(N-1)}{\eta_P(N)},$$

or equivalently the equations

$$\eta_a(N) = Q^{N-1} \frac{\eta_P(N+1)}{\eta_P(N)}, \quad \frac{1}{\eta_a(N-1)} = \frac{Q^{-N+2} \eta_P(N-1)}{\eta_P(N)}. \quad (29)$$

Again the latter two are not independent, but inverse of each other.

At last, from (28) and (29) by excluding η_a we infer the relation connecting $\eta_X(N)$ with $\eta_P(N)$, namely

$$\frac{\eta_X(N+1)}{\eta_X(N)} = Q^{2N+1} \frac{\eta_P(N+1)}{\eta_P(N)}. \quad (30)$$

Thus, for finding $\eta_a(N)$, $\eta_X(N)$ and $\eta_P(N)$ we have three relations: that is Eq.(30) and, say, the first ones in (28), (29), so that any two of the three are independent.

Now let us examine different possible situations.

- (i) It follows from (30) that $\eta_X(N) \neq \text{const} \cdot \eta_P(N)$ for any $Q \neq 1$.
- (ii) If $\eta_X(N)$ is known (or chosen), then $\eta_a(N)$ follows explicitly, see (28), and for $\eta_P(N)$ we have recursion relation which can be easily solved.
- (iii) Likewise, if $\eta_P(N)$ is known (or chosen), then $\eta_a(N)$ follows explicitly, see (29), and for $\eta_X(N)$ we have recursion relation which can be easily solved.

(iv) If $\eta_a(N)$ is fixed (chosen), then we have two similar, though not identical, recursion relations for $\eta_X(N)$ and $\eta_P(N)$ to be solved.

It is worth to consider some particular cases:

(a) Put $\eta_a(N) = Q^{-N-2}$ in (28). Then $\eta_X(N) = \text{const}$ and thus X is Hermitian: $X^\dagger = X$. For $\eta_P(N)$, from recurrence relation (29) we then find $\eta_P(N) = Q^{-N^2}$.

(b) Put $\eta_a(N) = Q^{N-1}$ in (29). Then $\eta_P(N) = \text{const}$ and thus P is Hermitian: $P^\dagger = P$. For $\eta_X(N)$, from recurrence relation (28) we then find $\eta_X(N) = Q^{N^2}$.

(c) Put $\eta_a(N) = 1$ that implies $(a^\pm)^\dagger = a^\mp$ (see also Sec. 6). Then from the respective recursion relations we find $\eta_X(N) = Q^{\frac{1}{2}N(N+3)}$ and $\eta_P(N) = Q^{\frac{1}{2}N(-N+3)}$.

(d) Let $\eta_a(N) = \text{const} \neq 1$, for instance $\eta_a = Q^\alpha$ with real α . Then for a^\pm we have standard pseudo-Hermitian conjugation of the shape $(a^\pm)^\dagger = Q^{\pm\alpha} a^\mp$. The remaining $\eta_X(N)$ and $\eta_P(N)$ are found from the relevant recurrence relations, and the result is $\eta_X(N) = Q^{\frac{1}{2}N(N+3\pm 2\alpha)}$ and $\eta_P(N) = Q^{\frac{1}{2}N(-N+3\pm 2\alpha)}$.

9 Commutation of X and P with the number operator N

For what follows we need the relations of permutation of the number operator N with the position or momentum operators,

$$[N, X] = \frac{1}{\sqrt{2}}(Q^{2N}a^+ - Q^Na^-) = X - 2Q^Na^-,$$

$$[N, P] = \frac{i}{\sqrt{2}}(Q^Na^+ + Q^{2N}a^-) = P + 2iQ^{2N}a^-,$$

from which we have

$$q^{\pm N}[N, X] \mp i[N, P] = -iP \pm q^{\pm N}X$$

and, denoting $q^{\pm N}X \equiv X_{N,q}^{(\pm)}$, infer

$$[N, X_{N,q}^{(\pm)} \mp iP] = \pm(X_{N,q}^{(\pm)} \mp iP) \Leftrightarrow N(X_{N,q}^{(\pm)} \mp iP) = (X_{N,q}^{(\pm)} \mp iP)(N \pm 1). \quad (31)$$

It is also possible to infer an interesting relations (containing a^- explicitly), e.g.

$$\begin{aligned} NX &= X(N+1) - \sqrt{2}q^Na^-, \\ N^2X &= X(N+1)^2 - \sqrt{2}q^Na^-(2N), \\ N^3X &= X(N+1)^3 - \sqrt{2}q^Na^-(3N^2+1), \\ N^4X &= X(N+1)^4 - \sqrt{2}q^Na^-(4N^3+4N), \end{aligned}$$

and so on. It is easily seen that these particular cases generalize to

$$N^kX = X(N+1)^k - \sqrt{2}q^Na^-A_k(N),$$

where $A_k(N)$ obeys the recurrence formula

$$A_{k+1}(N) = 2NA_k(N) - (N-1)(N+1)A_{k-1}(N)$$

solved by

$$A_k(N) = \sum_{r=0}^{k-1} (N+1)^{k-1-r} (N-1)^r = \frac{(N+1)^k - (N-1)^k}{2}. \quad (32)$$

Equivalently,

$$N^k X = X(N+1)^k - \sqrt{2}q^N A_k(N+1)a^-. \quad (33)$$

Using the latter, we arrive at the desired relation involving general function $\mathcal{F}(N)$:

$$\mathcal{F}(N)X = X\mathcal{F}(N+1) - [\mathcal{F}(N+2) - \mathcal{F}(N)]\tilde{a}^-, \quad \tilde{a}^- \equiv \frac{1}{\sqrt{2}}Q^N a^-. \quad (34)$$

Likewise, for the pair N and P we first obtain (compare with (33))

$$N^k P = P(N+1)^k + i\sqrt{2}q^{2N} A_k(N+1)a^-, \quad (35)$$

with the same $A_k(N)$ as in (32) above. Again, from the latter formula we find for general function $\mathcal{F}(N)$ the relation

$$\mathcal{F}(N)P = P\mathcal{F}(N+1) + [\mathcal{F}(N+2) - \mathcal{F}(N)]\hat{a}^-, \quad \hat{a}^- \equiv \frac{i}{\sqrt{2}}Q^{2N} a^-. \quad (36)$$

Note that for particular $\mathcal{F}(N) = Q^N$, or Q^{-N} the above formulas take simpler form:

$$\begin{aligned} Q^{\pm N} X &= XQ^{\pm(N+1)} \pm (1 - Q^2)Q^{\pm(N+1)-1}\tilde{a}^-, \\ Q^{\pm N} P &= PQ^{\pm(N+1)} \mp (1 - Q^2)Q^{\pm(N+1)-1}\hat{a}^-. \end{aligned}$$

We see that under the replacement $N \rightarrow N \pm 1$ the entities \tilde{a}^- and \hat{a}^- in these formulas do not change. Remark also that $a^-\tilde{a}^- = Q\tilde{a}^-a^-$, $a^-\hat{a}^- = Q^2\hat{a}^-a^-$. Using the above results (34) and (36) we deduce the following relation of permutation

$$\mathcal{F}(N)(X_{N,q}^{(\pm)} \mp iP) = (X_{N,q}^{(\pm)} \mp iP)\mathcal{F}(N \pm 1) \quad (37)$$

for an operator function $\mathcal{F}(N)$ (possessing expansion into a formal series). This is nothing but generalization of Eq. (31).

Let us stress again that the obtained relations of commutation between X , P and (a function of) N , see (34), (36) and (37), are of importance just for the chosen (in ref. [20] and herein) line of research based on the link: deformed Heisenberg algebra \Leftrightarrow deformed oscillator algebra. That will be used in our subsequent work.

10 Hamiltonian in terms of the position and momentum operators

Consider first the particular case $\mu = 0$ of the algebra (10). We use the Hamiltonian taken in the conventional form [35]

$$\mathcal{H} = \frac{1}{2}(aa^+ + a^+a) = \frac{1}{2}\left(\tilde{\Phi}(N+1) + \tilde{\Phi}(N)\right) \quad (38)$$

which yields the energy spectrum $E(n) = \frac{1}{2}(\tilde{\Phi}(n+1) + \tilde{\Phi}(n))$ in the Fock-like basis. With account of eq. (9) and recalling that $d_N \equiv d_{N,Q} \equiv \sqrt{2}(1 + Q^{2N})^{-1}$, we find the Hamiltonian in terms of the position and momentum operators, namely

$$\mathcal{H} = \frac{1}{2}d_N Q^{-N} \{ (d_{N+1} + Qd_{N-1})(X^2 + Q^{-1}P^2) + \\ + i(Q^N d_{N+1} - Q^{1-N} d_{N-1})PX + i(Q^N d_{N-1} - Q^{-1-N} d_{N+1})XP \} , \quad (39)$$

which is somewhat reminiscent of the Swanson model [37]. With the use of (4) this Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2}d_N Q^{-N} \{ (d_{N+1} + Qd_{N-1})[X^2 + Q^{-1}P^2 + iQ^{-1}(Q^N - Q^{-N})XP + \\ + (1/q)(Q^N d_{N+1} - Q^{1-N} d_{N-1})] \} \quad (40)$$

with PX term now absent. Note that at $p \rightarrow 1$ the results obtained here for the p, q -deformed HA reduce to those of the one-parameter case (since (4) reduces to the q -deformed HA considered in Ref. [17]), whereas for the case $Q = 1$ and $p = q \neq 1$ we come to the structure function $\phi(n) = \frac{n}{q}$, with X and P the same as those mentioned in the last line of Sec.3. Obviously, that again leads to the usual harmonic oscillator, whose spacing in the (linear) energy spectrum gets $\frac{1}{q}$ -scaled.

Hermiticity of the Hamiltonian

As mentioned our Hamiltonian has the form $\mathcal{H} = \frac{1}{2}(a^- a^+ + a^+ a^-)$, see (38). Recall that the creation and annihilation operators are in general not Hermitian conjugates of each other but instead satisfy the rules of generalized mutual $\eta(N)$ -pseudo-Hermitian conjugation, see Eq. (14) or Eq. (26). However, in view of (16) and (17) this form of Hamiltonian guarantees that it is Hermitian. The same is true for (39) and (40) as these are related with (38) through simple transformation.

The Hamiltonian \mathcal{H} , with account of the equality

$$\frac{p}{2}Q^{2N+1}(1 + Q^{2N+2})a^- a^+ - \frac{p}{2}Q^{2N}(1 + Q^{2N-2})a^+ a^- = 1,$$

see Eq.(7) and the formulas above Eq.(8), can be presented as

$$\mathcal{H} = \frac{1}{p} \frac{Q^{-2N-1}}{1 + Q^{2N+2}} + \frac{1}{2} \left(1 + Q^{-1} \frac{1 + Q^{2N-2}}{1 + Q^{2N+2}} \right) a^+ a^- . \quad (41)$$

This is still Hermitian, in view of Hermiticity of (an operator function of) N and the property (16) of $a^+ a^-$. At $p = q$, we have $a^- a^+ - a^+ a^- = q^{-1}$ and $\mathcal{H} = \frac{1}{2q} + a^+ a^-$. When $q = 1$, the usual harmonic oscillator with $\mathcal{H} = \mathcal{H}_0 = \frac{1}{2} + a^+ a^-$ is recovered.

Remark 4. The versions of Hamiltonian \mathcal{H} given in (39), (40) and (41) are equivalent to the initial one (38) and thus as well Hermitian. On the other hand, the form of Hamiltonian $H = \frac{1}{2}(X^2 + P^2)$, i.e. the standard one for harmonic oscillator, is not plausible, being neither Hermitian nor pseudo-Hermitian in the deformed case of $Q \neq 1$. We can however *suggest natural and simple modification* of H given in terms of η_X -pseudo-Hermitian X and η_P -pseudo-Hermitian P :

$$\tilde{\mathcal{H}} = \frac{1}{2} \left((\eta_X)^{-\frac{1}{2}} X^2 (\eta_X)^{\frac{1}{2}} + (\eta_P)^{-\frac{1}{2}} P^2 (\eta_P)^{\frac{1}{2}} \right). \quad (42)$$

Then, with the particular η_X and η_P e.g. corresponding to $\eta_a = 1$, see case (c) at the end of Sec. 8, we obtain

$$\tilde{\mathcal{H}} = \frac{1}{2} \left(Q^{-\frac{1}{4}N(N+3)} X^2 Q^{\frac{1}{4}N(N+3)} + Q^{\frac{1}{4}N(N-3)} P^2 Q^{-\frac{1}{4}N(N-3)} \right). \quad (43)$$

One can easily check Hermiticity of (42) and (43). Note that if $Q \rightarrow 1$, then $\tilde{\mathcal{H}} \rightarrow H = \frac{1}{2}(X^2 + P^2)$.

Remark 5. Returning to the skew-hermiticity of Eq. (4), especially its l.h.s., as discussed in the last part of Sec. 3, we may state the following: since the Hamiltonian is Hermitian, and $\mu \in \mathbb{R}$, all the conclusions made at the end of Sec. 3 extend completely to the (skew-Hermiticity of) *three-parameter deformation* of the Heisenberg algebra, with its p, q, μ -deformed basic relation Eq. (10).

Discussion

In the present paper, for the 2- and 3-parameter extensions [20] of the Heisenberg algebra, assuming either usual, or generalized (with $\eta_a(N)$ involved) conjugation properties of a^- and a^+ , we studied the special non-Hermiticity of X, P , realized exactly in terms of the notion of $\eta_X(N)$ -pseudo-Hermiticity of X or/and $\eta_P(N)$ -pseudo-Hermiticity of P . Generally speaking, our main results concern precise and fully-coordinated (mutual or self-) conjugation properties of the four involved operators, with the crucial N -dependence of the eta-functions η_a and η_X, η_P . Let us stress once again the basical aspect that such N -dependence is caused by the important link earlier established in Ref. [20], namely: deformed Heisenberg algebra \Leftrightarrow deformed oscillator algebra. Also it is worth to note that the "metric" operators $\eta_a(N), \eta_X(N)$ and $\eta_P(N)$ are all Hermitian, since they are given as the corresponding functions of the Hermitian particle number operator.

The Hamiltonian \mathcal{H} in our treatment is Hermitian as it is formed from the bilinears a^+a^- and a^-a^+ . Remark that these bilinears are Hermitian, although the individual a^+ and a^- may be not (mutual) Hermitian conjugates, but rather the $\eta_a(N)$ -pseudo-Hermitian conjugates of one another. In addition we have also introduced the Hamiltonian $\tilde{\mathcal{H}}$ as given in Eqs. (42)-(43), and this presents yet another Hermitian deformation of the well-known Hamiltonian $H = \frac{1}{2}(X^2 + P^2)$ of harmonic quantum oscillator.

In a forthcoming work we intend to examine the spectra (eigenvalues, eigenfunctions) of the position and momentum operators, along with the Hamiltonian (43), *in the framework of coordinate realization*. On the other hand, when exploiting the (deformed) Fock like basis, the energy spectrum of the Hamiltonian $H = \frac{1}{2}\{a^+, a^-\}$ is explicitly known: namely it is given, see Eq. (38), through the respective structure function such as e.g. $\tilde{\Phi}(n)$ in Sec. 4. It is also worth to emphasize the importance to find and explore particular quantum physical systems governed by Hamiltonians such as (39) and alike, with the $\eta(N)$ -pseudo-Hermitian position and/or momentum operators as those studied above. Also, it would be interesting to compare such results with those obtained in the Swanson model [37]. These aspects, along with the study of $\eta(N)$ -pseudo-Hermitian Hamiltonians, will be among our nearest tasks.

Acknowledgement

This work was partially supported by the Special Programme of Division of Physics and Astronomy of NAS of Ukraine.

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